Ambiguity Functions for Spatially Coherent and Incoherent Multistatic Radar

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Abstract

The ambiguity function is a key tool for determining the target resolution capability of a radar system. Recently, multistatic radar systems have been proposed where target detection is jointly performed on a vector of captured signals arising from multiple spatially dispersed transmitters and/or receivers. In this paper, expressions for the ambiguity functions of such systems are derived based on corresponding statistically optimal multistatic detectors, which themselves depend on the spatial coherence of target fluctuations observed by each receiver. These expressions allow the ambiguity in resolving target position and velocity vectors in two or three dimensions to be determined for any transmitted waveform and multistatic topology. New plots are introduced to illustrate the resulting ambiguity responses by example, and the dependency on spatial, target and waveform parameters is discussed.
I. Introduction

In recent years there has been renewed interest in multistatic radar – that is, systems comprising multiple, spatially dispersed transmitting and/or receiving stations. This has been driven by requirements for increased coverage, improved classification (i.e. target information, including size, structure and relative movement) and greater accuracy of parameter estimation, in particular target position co-ordinates and velocity vectors specified in three dimensions. In addition, multistatic radars have greater tolerance to electronic countermeasures and sources of interference due to their spatial diversity, and the potential for improved physical survivability due to the multiplicity of stations [1]. Recent large-scale multistatic radar development programs include heterogeneous sensor networking to improve tracking and classification performance as part of the US Navy Cooperative Engagement Capability [2] and distributed aperture sensing using a cluster of micro-satellites for geolocation and imaging under the US Air Force Research Laboratory TechSat21 initiative [3].

Traditionally, multistatic signal processing has taken the form of post-detection data fusion, for example the formation of tracks based on individual plots obtained from several radars in a loosely connected system surveying a common coverage area [4]. However, the availability of low-cost digital signal processing and wide bandwidth data networks means that pre-detection fusion of data at the signal level is viable. In this case, a detection decision can be made without prior loss of information following the joint processing of all received signals. Therefore, some degree of spatial synchronization must exist across the multistatic system, which at a minimum involves time synchronization of events between all transmitters and receivers, and may also include
mutual phase coherency of the oscillators in all signal paths.

The ambiguity function is well-known in the context of monostatic radar as a key tool for determining target resolution capability, and is a consequence of the nature of the optimal detector, which involves decision-making based on the output of a matched filter determined from the transmitted waveform [5]. In this paper, following a brief review of the statistical basis for optimal detection and resulting ambiguity in monostatic and bistatic radar, ambiguity functions are derived for multistatic radar where the input to the detector is a vector comprising the signals from all receivers. Two distinct scenarios are considered that differ according to the spatial coherence of target fluctuations as observed by each receiver. The approach taken provides a generic framework that allows both closely and widely spaced multistatic topologies to be correctly analyzed for different classes of target. As a result, ambiguity responses can be obtained that allow the true performance of a multistatic system to be analyzed - specifically, the ability of the radar to resolve two- or three-dimensional target position and velocity vectors, compared to the monostatic matched filter detector where only range and radial velocity can be resolved (independently of the antenna beam pattern). Several contrasting topologies are modeled as examples and the resulting ambiguity responses are presented using a pair of new plots as an alternative to the traditional ambiguity diagram that clearly demonstrate the system performance. Finally, these results are discussed in terms of the dependency of the ambiguity response on the topology and the spatial nature of target fluctuations.

II. Monostatic and Bistatic Ambiguity Functions

This section briefly reviews background on monostatic and bistatic ambiguity functions
to provide context for the multistatic case. In the case of monostatic pulsed radar, the complex received signal arising from a slowly fluctuating point target at range

\[ R = ct_a / 2 \]

moving with constant radial velocity \( v = \Omega_a \lambda / 2 \) can be written as:

\[
X(t) = a \exp(j\varphi) s_0(t - t_a) \exp[j(\omega_0 + \Omega_a)(t - t_a)] + n(t)
\]

(1)

where \( \omega_0 \) is the angular carrier frequency, \( t_a \) is the propagation delay, \( \Omega_a \) is the apparent Doppler angular frequency and \( s_0(t) \) is the complex base-band waveform of the transmitted pulse. The waveform is assumed narrowband so that 'stretching' due to the target movement is insignificant. The complex amplitude \( a \exp(j\varphi) \) is assumed to be a zero-mean slowly varying complex Gaussian random variable that is approximately constant over a single observation period (Swerling I target). Therefore, the real magnitude \( a \) has a Rayleigh distribution, while the phase \( \varphi \) has a uniform distribution over \( -\pi \leq \varphi \leq \pi \). The complex noise process \( n(t) \) is assumed to be zero-mean, white and Gaussian. It is well known that the optimal detector for this signal under the Neyman-Pearson criterion is the thresholded matched filter or correlation filter, which may be written as:

\[
\Lambda_{mono} = \left| \int_{T/2}^{T/2} X(t) s_0^*(t - t_h) \exp[-j(\omega_0 + \Omega_h)(t - t_h)] dt \right|
\]

(2)

where \( t_h \) and \( \Omega_h \) are the matched or ‘hypothesized’ values of propagation delay and Doppler frequency respectively. The optimality of this detector may be proven by analysis of the likelihood ratio test using the Karhunen-Loève expansion [6]. It can be shown to maximize both the signal-to-noise ratio of \( \Lambda \) and, more importantly, the probability of detection given a fixed probability of false alarm; the threshold value for \( \Lambda \) should be chosen according to the desired balance between these two probabilities.
The ambiguity function \( \chi \) is defined as the normalized expectation of the output of this optimal detector (prior to thresholding), which is found by substituting the received signal in Equation 1 into Equation 2, disregarding the noise process and phase terms rendered irrelevant by the modulus and normalizing the amplitude to give:

\[
\chi_{\text{mono}}(\omega) = \left| \int_{-\infty}^{\infty} s_0(t-t_a)s_0^*(t-t_h)\exp[-j(\Omega_h - \Omega_a)t]dt \right|
\]  

This equation describes the response of the detector matched to \( \{t_h, \Omega_h\} \) when the actual received signal has \( \{t_a, \Omega_a\} \) delay and Doppler frequency respectively. As the response of the detector for relative delay and Doppler frequency \( \tau = t_h - t_a \) and \( \Omega = \Omega_h - \Omega_a \) is independent of the actual values \( \{t_a, \Omega_a\} \), it may be expressed in a well-known form that depends only on the complex base-band transmitted waveform \( s_0(t) \):

\[
\chi(\tau, \Omega)_{\text{mono}} = \left| \int_{-\infty}^{\infty} s_0(t)s_0^*(t-\tau)\exp(-j\Omega t)dt \right|
\]  

The independent variables delay and Doppler are linearly proportional to range and radial velocity respectively.

The ambiguity function directly determines the capability of a radar system to resolve two targets that exist at different ranges from the radar and/or have different radial velocities. When the received signals from the targets have similar energy, the nominal resolution is equal to the half-power width of the ambiguity function mainlobe [7]. On the other hand, when the received signal from one target is much larger than that from the other, the ambiguity function determines the extent to which, for a given difference in range and radial velocity \( (\tau, \Omega) \), the sidelobes of the response from one target will
obscure the other target. The ambiguity function is also related to the accuracy with which the range and radial velocity of a given target can be estimated. When the SNR is high, it has been shown that the Cramér-Rao lower bounds on estimation accuracy (so-called ‘local accuracy’) are dependent on both the SNR and the second derivatives of the ambiguity function [8] – in other words, the sharpness of the ambiguity function mainlobe.

The application of the ambiguity function to bistatic radar has been previously considered by Tsao et al [9]. The same matched filter detector is optimal, but the relationships between the fundamental independent variables delay/Doppler and the parameters of interest (range and velocity) are non-linear due to the bistatic topology, and can be written in vectorial form as:

$$t_a = \frac{(|\mathbf{R}_T| + |\mathbf{R}_R|)}{c}$$  \hspace{1cm} (5)

$$\Omega_a = \frac{\left(\frac{\mathbf{R}_T \cdot \mathbf{V}}{|\mathbf{R}_T|} + \frac{\mathbf{R}_R \cdot \mathbf{V}}{|\mathbf{R}_R|}\right)}{\lambda}$$

where $\mathbf{R}_R$ and $\mathbf{R}_T$ are the distance vectors from the target to the transmitter and receiver respectively, $\mathbf{V}$ the velocity vector of the target, and $\cdot$ represents the vector dot-product.

Therefore, it is no longer appropriate to use the standard delay/Doppler ambiguity diagram as it does not demonstrate the effective ambiguity related to these parameters. Instead, it was proposed that the function described by Equation 3 be plotted in terms of the range from the receiver (in the direction towards the actual target position) and the component of target velocity in the direction of the bistatic bisector, according to the standard bistatic ‘North’ axes [10]. In doing so, the non-linear change of variables ‘warp’ the axes of the standard ambiguity function according to the relative topology of the
transmitter, receiver and target, such that the absolute hypothesized and actual range/velocity values as well as the transmitted carrier frequency must be specified to determine the ambiguity response. Therefore, while losing some utility as a general method for determining radar performance based only on the transmitted waveform, the bistatic ambiguity function is a useful tool for the difficult task of predicting system response for a given bistatic topology.
III. Signal model for multistatic radar

The derivation of ambiguity functions for multistatic radar firstly requires the development of suitable models for the received signals at each receiver, from which optimal detectors for the joint processing of a vector of these received signals can be determined, incorporating a generalization of the spatial dependencies outlined above for the bistatic case.

A multistatic radar system comprising a single transmitter and $m$ receivers is considered. The vector $\mathbf{X}(t) = \mathbf{S}(t) + \mathbf{N}(t)$ is defined where each element $X_i(t)$ is the actual time-varying signal at receiver $i$, $\mathbf{S}$ is the vector of 'wanted' signals, and $\mathbf{N}$ is a vector of mutually uncorrelated, zero-mean white Gaussian random noise processes at each receiver.

Therefore, following Equation 1, the elements of $\mathbf{X}$ are given by:

$$X_i(t) = a_i \exp(j\varphi_i)s_0(t - t_{ai})\exp[j(\omega_0 + \Omega_{ai})(t - t_{ai})] + n_i(t)$$

where $t_{ai}$ is the propagation delay from the transmitter to the $i$th receiver via the target, $\Omega_{ai}$ the apparent Doppler angular frequency and $a_i \exp(j\varphi_i)$ the complex amplitude observed by the $i$th receiver. The latter term is particularly important as it incorporates propagation loss and signal fluctuations, which may be considerably different for each receiver.

In practice, signal fluctuations may be caused by small relative movements of a target comprising multiple scattering points, which change its apparent instantaneous complex RCS (other possible causes include variations in the propagating medium and interference due to non-stationary multipath clutter). In monostatic and bistatic radar, where there is just a single look angle, only the temporal statistical nature of the
fluctuations is of concern; the zero-mean complex Gaussian model (Swerling I target) assumed in Section II is suitable if the number of scattering points is large and each has similar effective area. In the case of multistatic radar, the spatial coherence of the fluctuations (i.e. their mutual correlation as observed by each receiver) is also important. This issue has been the subject of some recent theoretical studies [11, 12, 13], and bounds have been proposed for the conditions under which fluctuations at each receiver can be expected to be mutually correlated or independent. Roughly speaking, it has been found that, for a Swerling I type target where the distance from the target to each of two receivers is approximately equal, the mutual correlation of the fluctuations observed at the receivers will be high only if the ratio of the target-receiver distance to the distance between receivers is much greater than the apparent size of the target in RF wavelengths. In other words, the fluctuations observed at each receiver due to a large complex target will be mutually independent unless the receivers are closely spaced, whereas those due to a small simple target may be highly correlated for both closely and widely spaced topologies. By definition, the deterministic received signals from a non-fluctuating target (e.g. an isotropic point scatterer or perfectly stationary target of arbitrary type in a controlled environment) are completely mutually correlated for any topology.

Furthermore, it is evident that the mutual correlation of the phase $\varphi_i$ of each received signal at base-band also depends on the spatial coherence of the multistatic radar system itself - specifically the local oscillators in signal paths across the entire distributed system must be mutually phase-locked if correlated target phase fluctuations are to be observed. Therefore, to simplify the consideration of a large number of system and target parameters in the derivation of multistatic ambiguity functions, two typical and
contrasting signal models are chosen as the basis for subsequent analysis, the definitions and validity conditions for which are described in Table 1.
Table 1: Definitions of 'coherent' and 'incoherent' signal models and conditions for validity

<table>
<thead>
<tr>
<th>Model</th>
<th>Nature of fluctuations</th>
<th>Conditions for validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>'Coherent'</td>
<td>All $a_i \exp(j\varphi_i)$ are zero-mean complex Gaussian random variables that are completely mutually correlated. Therefore all $a_i$ are related to the real signal magnitude at (arbitrary reference) receiver $1$ through a deterministic scaling factor $A_{i1} = a_i / a_1$, and all $\varphi_i$ are related to the signal phase at receiver $1$ through a deterministic phase offset $\Delta\varphi_{i1} = \varphi_i - \varphi_1$.</td>
<td>The multistatic system itself is spatially coherent, and one of the following is true: (i) the target is fluctuating with Swerling I model (between observations), and the receivers are adequately closely spaced such that the observed fluctuations at each are mutually correlated; (ii) the target is non-fluctuating during the complete period of all observations; the random variable $a_i \exp(j\varphi_i)$ can be considered to be temporally correlated over this period such that $a_i$ and $\varphi_i$ are effectively constant.</td>
</tr>
<tr>
<td>'Incoherent'</td>
<td>All $a_i \exp(j\varphi_i)$ are mutually independent zero-mean complex Gaussian random variables.</td>
<td>The multistatic system itself has arbitrary spatial coherence, the target is fluctuating between observations with Swerling I model and the receivers are adequately widely spaced to result in mutually uncorrelated fluctuations.</td>
</tr>
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</table>
IV. Optimal multistatic detectors

To determine ambiguity functions for multistatic radar based on the signal models described in Section III, it is firstly necessary to determine the corresponding optimal detectors. The derivation of such detectors for a vector of received signals from a multistatic radar with a single transmitter and $m$ receivers has been previously considered by Conte et al [14] and Chernyak [13] and is summarized below.

As per the monostatic case, the optimal Neyman-Pearson detector is formed from the likelihood ratio, which is the ratio of the probability density functions under the hypotheses that the expected signal vector $S$ is present in, and absent from, the received signal vector $X$. The likelihood ratio may be written as [6]:

$$
\Lambda = \exp \left\{ \Re \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} S^*(t_1) R(t_1, t_2) X(t_2) dt_1 dt_2 - \frac{1}{2} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} S^*(t_1) R(t_1, t_2) S(t_2) dt_1 dt_2 \right\}
$$

where $\Lambda$ is the single output variable from which detection decisions can be made by binary thresholding, and matrix $R$ is the inverse of the space-time covariance matrix of the noise vector $N$. Since elements of $N$ are assumed to be mutually independent white zero-mean Gaussian random processes, $R$ is diagonal with non-zero elements equal to $1/N_{ii}$ where $N_{ii}$ is the noise spectral density at the $i$th receiver.

Elements of the vector of expected signals $S$ in Equation 7 are given by:

$$
S_i(t) = a_i \exp(j\varphi_i) s_0(t - t_{hi}) \exp[j(\omega_0 + \Omega_{hi})(t - t_{hi})]
$$

where, in common with the monostatic detector in Equation 2, subscript $h$ refers to the values of delay and Doppler hypothesized by the detector (rather than those of the actual received signal); the likelihood ratio test being optimal under the conditions $t_{hi} = t_{ai}$ and $\Omega_{hi} = \Omega_{ai}$, for all $i$. Since the received signals are assumed to be randomly fluctuating,
the expected amplitude and phase terms \(a_i\) and \(\varphi_i\) are a-priori unknown. Therefore, the likelihood ratio of Equation 7 is conditional on fixed values of \(a_i\) and \(\varphi_i\), and the unconditional form must be found by integrating over these random parameters.

In the case of the coherent signal model defined in Section III, substituting the constant amplitude scaling factors \(A_{si} = a_i / a_i\) and phase offsets \(\Delta \varphi_{si} = \varphi_i - \varphi_i\) into Equation 7, integrating over \(a_i\) and \(\varphi_i\) with their known probability density functions (Rayleigh and uniform respectively) and reducing to the sufficient statistic gives the detection equation [13]:

\[
\Lambda_{coherent} = \left[ \sum_{i=1}^{m} \frac{A_{si}^{2}}{N_i} \exp(-j \Delta \varphi_{si}) \int_{-T/2}^{T/2} X_i(t) s_0^*(t - t_{hi}) \exp[-j(\omega_0 + \Omega_{hi})t] dt \right]^2
\]  

(9)

In the case of the incoherent signal model where all \(a_i\) and \(\varphi_i\) have the same probability density functions but are statistically independent, the optimal detector is found by integrating over all these random parameters giving:

\[
\Lambda_{incoherent} = \sum_{i=1}^{m} \left\{ \frac{\overline{A}_{si}^{2}}{N_i^2 (1 + \overline{E}_i / N_i)} \int_{-T/2}^{T/2} X_i(t) s_0^*(t - t_{hi}) \exp[-j(\omega_0 + \Omega_{hi})t] dt \right\}^2
\]  

(10)

where \(\overline{A}_{si}^{2}\) is the mean signal power compared to receiver 1 and \(\overline{E}_i\) is the absolute mean signal energy at receiver \(i\).

In both the above detectors, the hypothesized delay and Doppler terms can be calculated from the target position and velocity to which the detector is matched by extension of the bistatic case in Equation 5, giving:

\[
t_{hi} = (|\mathbf{R}_T| + |\mathbf{R}_R|) / c
\]  

(11)
\[
\Omega_{hi} = -\left( \frac{R_T \cdot V}{|R_T|} + \frac{R_{R_i} \cdot V}{|R_{R_i}|} \right) / \lambda
\]

where \( R_{R_i} \) is the distance vector between the hypothesized target and the \( i \)th receiver.

A comparison of Equations 9 and 10 reveals that both detectors are based on the summation of the outputs of individual matched filters operating on the signals at each receiver (compare with Equation 2). Each of these filters is matched to hypothesized target position and velocity vectors defined in three-dimensions which are converted to equivalent delay/Doppler values \( \{ \Omega_{hi}, \Omega_{hi} \} \) through Equation 11. In general, these non-linear relationships are different for each matched filter component according to the individual geometries of each bistatic transmitter-receiver pair comprising the multistatic system. In effect, the detectors perform delay equalization (and, in the coherent case, phase equalization) that results in alignment of all received signals if a target exists at the hypothesized position.

In the case of the coherent signal model detector in Equation 9, the output value \( \Lambda_{\text{coherent}} \) equals the modulus of the coherent vectorial sum of the output of these individual matched filters, which are amplitude weighted according to the corresponding relative signal amplitude to noise power ratio \( A_{r_i} / N_i \), and phase shifted by \( \Delta \varphi_{r_1} \). A detection decision can then be made in the usual way by binary thresholding \( \Lambda \) according to the required detection and false-alarm probabilities. The amplitude weighting component of this detector is in fact equivalent to that used with monostatic radar for optimal integration of multiple pulses with different amplitudes [15] as well as by the ‘maximal ratio combiner’ (MRC) diversity technique in wireless communications [16], all of which maximize the SNR of the summed output, which it can be shown is equal to the total of
the SNR of all signals comprising the summation.

On the other hand, in the case of the incoherent signal model detector in Equation 10, the output value $\Lambda_{\text{incoherent}}$ equals the (incoherent) sum of the square-modulus of the output of the individual matched filters, amplitude weighted according to the relative average signal power $\overline{A_i^2}$ and actual signal energy $E_i$. Since each phase $\varphi_i$ also fluctuates independently, coherent gains cannot be achieved, and the phase is discarded by taking the square-modulus before summation.

It can be seen from this analysis that the choice of signal model considerably affects the form of the optimal detector itself. Therefore, unlike the bistatic ambiguity function where only the transmitted waveform and topology are relevant, a multistatic ambiguity function defined as the response of an optimal detector cannot be decoupled from its dependency on the target model.

V. Single-transmitter-multiple-receiver ambiguity functions

Having defined the optimal multistatic detectors for the coherent and incoherent signal models, the corresponding ambiguity functions can now be derived which, as per the monostatic case, are defined as the normalized expectation of the output of the detectors prior to thresholding.

For the coherent detector, substituting the received signal (Equation 6) into Equation 9, noting that $A_i a_i = \overline{A_i^2} a_i$ where $a_i$ should be replaced by its expectation or mean value (a constant), disregarding the noise process and normalizing the amplitude yields the coherent ambiguity function [17]:
\[
X_{\text{coherent}} = \left( \sum_{i=1}^{m} A_i^2 \right) \sum_{i=1}^{m} \left( \frac{A_i^2}{N_i} \right) \exp \left[ j(\omega_i(t_i - t_{ai}) + \Omega_i(\tau_{hi} - \Omega_{hi}t_{ai})) \right] \left[ \int_{-\tau_i/2}^{\tau_i/2} s_0(t - t_{ai}) s_0^*(t - t_{ai}) \exp \left[ j(\Omega_{ai} - \Omega_{hi})\tau \right] d\tau \right]
\]

(12)

For the incoherent detector, substituting the received signal into Equation 10 and replacing \( a_i \) with its mean value \( \bar{a}_i \), noting the identity \( |x|^2 = xx^* \) and that \( \bar{A}_i^2 \bar{a}_i^2 = \bar{A}_i^2 \bar{a}_i^2 \), yields the normalized incoherent ambiguity function:

\[
X_{\text{incoherent}} = \left( \sum_{i=1}^{m} \frac{\bar{A}_i^4}{N_i^2 (1 + E_i / N_i)} \right) \sum_{i=1}^{m} \left( \frac{\bar{A}_i^4}{N_i^2 (1 + E_i / N_i)} \right) \left[ \int_{-\tau_i/2}^{\tau_i/2} s_0(t - t_{ai}) s_0^*(t - t_{ai}) \exp \left[ j(\Omega_{ai} - \Omega_{hi})\tau \right] d\tau \right]^2
\]

(13)

The actual delay and Doppler values \( \{t_{ai}, \Omega_{ai}\} \) can be mapped to the multistatic geometry using Equation 11 with the real position and velocity of the target.

Analysis of Equations 12 and 13 gives some revealing insights into the nature of the ambiguity response for the two cases. The ambiguity function for the coherent signal model in Equation 12 is formed from the coherent sum of baseband bistatic ambiguity functions (compare with Equation 3, before the modulus is taken) mapped to the multistatic geometry, each of which is multiplied by a phase term that is dependent on the RF frequency as well as the actual and hypothesized delay and Doppler values, before being amplitude weighted. As such, the complete system of receivers may be considered to be a highly distributed antenna array. Indeed, in the extreme case where all receivers are positioned in a close linear topology separated by \( \lambda / 2 \), it can be shown that the positional ambiguity response reduces to that of a monostatic radar transmitting the same waveform and using a ‘true time delay’ digital array beamformer. However, in the general case where the antennas are arbitrarily and widely dispersed, the hypothesized target positions generally exist within the nearfield of the effective antenna system, and the positional ambiguity response of the detector focuses to a particular point in space rather
than forming a continuous beam. This localization capability derives not from the individual antenna beam patterns at each receiver but is intrinsic to the multistatic detection process. In comparison, a monostatic radar with an isotropic antenna can only resolve in range, so it is completely ambiguous in terms of localization over any iso-range spherical surface.

The ambiguity function for the incoherent signal model in Equation 13 is formed from an amplitude weighted power sum of bistatic ambiguity functions, so the equivalence to a distributed array processor is no longer valid, and the nature of the ambiguity response can be expected to be quite different to the coherent case.

In summary, the multistatic ambiguity functions have a complex dependency on the multistatic topology, hypothesized and actual target parameters as well as the transmitted signal. In the coherent case, the spatial information encoded in the phase terms involving the RF carrier, which are negated by the modulus in the monostatic ambiguity function of Equation 3, are maintained within the summation and give rise to a spatial focusing capability equivalent to that of a highly distributed array processor.

**VI. Multiple-transmitter-multiple-receiver ambiguity functions**

The ambiguity functions derived in Section V can be extended to the case of $n$ transmitter, $m$ receiver multistatic radar. A typical implementation is considered where the waveforms assigned to each transmitter form a mutually orthogonal set for all possible relative time delays - in other words, the cross correlation of any two discrete-time waveforms is approximately zero for any lag [18]. This simplifies the analysis since the transmitted signals can be separated at each receiver using a bank of
The coherent signal model described in Section III is extended such that the fluctuations associated with all \( mn \) partial signals (\( n \) transmitted waveforms observed at \( m \) receivers) are mutually correlated. Likewise, the incoherent signal model is extended such that all \( mn \) partial signals fluctuate independently. For a complex fluctuating target, the coherent model is expected to be valid when all transmitters and receivers are closely spaced, while the incoherent model is valid when they are all widely dispersed. Then, the corresponding ambiguity functions are found from the optimal detectors by extending Equations 9 and 10 to sum over the outputs of all \( mn \) matched filters, resulting in:

\[
\chi_{\text{coherent}MN} = \left( \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{A_{ij1}^2}{N_j} \right)^{-1} \sum_{j=1}^{n} \sum_{i=1}^{m} \left\{ \frac{A_{ij1}^2}{N_j} \exp \left[ j(\omega_j(t_{hij} - t_{aij}) + \Omega_{hij} - \Omega_{aij} t_{aij}) \right] \right\} \times \\
\int_{T/2}^{T/2} s_{0j}(t - t_{aij}) s_{0j}^*(t - t_{hij}) \exp \left[ j(\Omega_{aij} - \Omega_{hij}) t \right] dt 
\]

\[
\chi_{\text{incoherent}MN} = \left( \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{\overline{A}_{ij1}^4}{(1 + E_j / N_j)} \right)^{-1} \sum_{j=1}^{n} \sum_{i=1}^{m} \left\{ \frac{\overline{A}_{ij1}^4}{(1 + E_j / N_j)} \times \right\} \\
\int_{T/2}^{T/2} s_{0j}(t - t_{aij}) s_{0j}^*(t - t_{hij}) \exp \left[ j(\Omega_{aij} - \Omega_{hij}) t \right] dt \times
\]

where \( A_{ij1} = a_j / a_{1j} \) is the relative amplitude of the signal from the \( j \)th transmitter as observed by the \( i \)th receiver compared to the signal from transmitter 1 observed at receiver 1, \( s_{0j} \) is the baseband waveform assigned to the \( j \)th transmitter, \( t_{aij} \) is the propagation delay from the \( j \)th transmitter to the \( i \)th receiver via the target and \( \Omega_{ai} \) the apparent Doppler shift angular frequency for that bistatic pair. As before, the subscript \( h \) refers to the hypothetical values to which the detector is matched rather than the actual values.
Multiple-transmitter multiple-receiver multistatic systems such as these, which are capable of processing received data at the signal level, have been referred to in some literature as 'MIMO radars' [19], on the basis of recent information theoretic analysis that has parallels with MIMO communications and demonstrates two potential advantages in radar performance.

First, if the multistatic topology is such that target fluctuations are independent, diversity gain can be achieved because fluctuation nulls are unlikely to occur simultaneously in the received partial signals corresponding to all \( mn \) bistatic pairs [20]. Therefore, although coherent SNR gains cannot be obtained, the probability of false-alarm may be reduced particularly if a very high probability of detection is required. In fact, this analysis is not in principle unique to MIMO radars since single-transmitter multiple-receiver multistatic systems may also be arranged such that fluctuations are independent, as per the incoherent signal model in Section III. However, since the diversity gain increases with the number of independently fluctuating signals, MIMO radar has the advantage that its \( mn \) bistatic pairs are greater in number than the \( m \) bistatic pairs in a single transmitter \(( n = 1)\) system, given a fixed total number of stations \( m + n \).

Second, if conversely the multistatic topology and target type are such that any fluctuations are correlated so the signals can be processed coherently, it has been suggested that 'spatial multiplexing' gains can be achieved in MIMO radars where the spatial diversity of the bistatic pairs enables high resolution target localization. Section VIII will show that such advantages also exist in single transmitter multistatic systems, as suggested in the analysis of the ambiguity function for the coherent signal model in Section V with its capability to coherently focus the signals from all distributed receivers.
to a particular point in space. On the other hand, MIMO radars have an additional degree of freedom in the set of transmitted signals $s_{0j}$ (waveform diversity), as well as additional spatial degrees of freedom in a topology involving both multiple transmitters and receivers, which might potentially be used to optimize the overall ambiguity response for particular applications. For example, a hybrid topology where transmitters are widely spaced and receivers are closely spaced has been proposed for achieving improved performance in the application of direction finding [21].
VII. Comparison with previous work

This section compares the equations derived in Sections V and VI to various works that relate to multistatic radar ambiguity in the recent literature. Bradaric et al [22] defined an ambiguity function for a single transmitter, multiple receiver system with the assumption that the complex amplitudes of each received signal are statistically independent, based on an optimal detector derived by Conte et al [14]. This corresponds to the incoherent signal model used in this paper, and the resulting ambiguity function is equivalent in form to that of Equation 13. The ambiguity response is determined as a function of range from the transmitter and speed in a fixed direction, rather than the general three-dimensional vectorial approach used in this paper, and the coherent case is not considered.

On the other hand, Lehmann et al [23] consider the case for MIMO radar observing a single non-fluctuating stationary point target. The appropriate detector is defined as an unweighted coherent sum of individual matched filters. Therefore, while the component of the phase term in Equation 14 that is responsible for resolution improvement due to coherent spatial processing is included, the detector is suboptimal in a Neyman-Pearson sense since amplitude weighting is not performed before summation, and Doppler shifts caused by target movement are not considered. A MIMO ambiguity function has also been derived by San Antonio et al [24], assuming a non-fluctuating point target with constant velocity. Again, the detector is defined as an unweighted coherent sum, so the optimal amplitude weighting is not included. The resulting ambiguity function contains phase terms involving the RF frequency which, by some rearrangement, can be shown to be equivalent to those in Equation 14. However, the resulting ambiguity responses are only analyzed for the far-field case where they can be plotted as a function of range and
angle, rather than the general approach used in this paper that also applies to the near-field of the complete system of antennas.

VIII. Illustrative examples

The nature of the ambiguity responses corresponding to both the coherent and incoherent optimal detectors is now demonstrated by way of parameterized examples. A single-transmitter multistatic radar with isotropic antennas is chosen for analysis (using Equations 12 and 13) so as to allow comparison with recent analysis of the MIMO case, and to clearly highlight the dependency on the spatial multistatic topology and target model.

Firstly, some simplifications are made to illustrate the nature of the ambiguity response with the minimum of parameters. In the coherent case, the relative signal to noise ratio $A_{ii}^2 / N_i$ at each receiver will in reality be dependent on the relative instantaneous target RCS and propagation loss for each bistatic pair, the noise figure of each receiver, and so on. These factors are ignored in monostatic ambiguity analysis as they do not change the normalized response; however, in the multistatic case they affect the weighting of the components from each receiver. Here, the simplification is made that $A_{ii}$ is invariant to observation geometry, and that all receivers are identical such that the only dependency is on diffractive energy loss due to propagation:

$$A_{ii}^2 / N_i \propto 1/(|R_i|^2 |R_{R_i}|^2)$$

In the incoherent case, to avoid dependency on the actual signal energy, it is additionally assumed that the received signals are weak ($\bar{E}_i < N_i$), such that the amplitude weighting term (based on mean signal power) in Equation 13 reduces to:
\[
\frac{\bar{A}_i^2}{N_i^2} \propto 1/(|R_{\tau}|^4 |R_{\nu}|^4)
\]

In the case of strong signals, the weighting performed by the detector may deviate slightly from this assumption; however, the general form of the response will not be substantially altered.

A typical radar specification is chosen and used for all the examples in this section; the transmitted signal \( s_0(t) \) is a train of five identical chirp pulses (5 \( \mu \)s pulse length, 50 MHz bandwidth) with 10 kHz PRF, and the RF carrier frequency is 10 GHz. A monostatic radar with this specification has nominal range and radial velocity resolution of approximately 3 m and 30 m/s respectively.

In accordance with the convention for plotting ambiguity diagrams based on power, ambiguity plots based on Equation 12 are squared (i.e. \( \chi^2_{\text{coherent}} \)), whereas those based on Equation 13 are plotted as-is since the multistatic summation is performed on a power (squared modulus) basis.

The transmitter/receiver positions and target position/velocity are defined on a two-dimensional plane for ease of illustration. With reference to an arbitrary origin at the bottom left of each figure, a single target is assumed to exist at position vector \( \{500,500\} \) meters, traveling at 20 m/s in the due South direction. Therefore, the ambiguity plots demonstrate the position and velocity resolution capability for this target by showing the response of the corresponding detector for all possible hypothesized or matched values.

For clarity, the axes limits are restricted to values close to the actual target parameters – therefore ambiguity peaks resulting from multiple-time-around delay and aliased Doppler (due to the PRF of the chosen waveform) are not visible.

Firstly, for purposes of comparison, Equation 12 is plotted for the monostatic topology
\((m=1)\) shown in Figure 1a, where the co-located transmitter and receiver are at position vector \(\{500,0\}\). The resulting traditional mesh and iso-response contour plots are shown in Figure 2. Points on the range axis refer to hypothesized positions on the line intercepting the radar and the actual target position; likewise points on the speed axis refer to the magnitude of hypothesized velocity vectors pointing towards the radar. Figure 2 differs from the conventional ambiguity diagram for \(s_o(t)\) only in that the peak response exists at the actual target parameters of 500 m range and 20 m/s velocity, rather than being normalized to the origin.

![Figure 1: (a) Monostatic and (b) bistatic topologies](image)

![Figure 2: Monostatic traditional ambiguity diagram (linear power response)](image)
The use of these traditional ambiguity diagrams for analyzing multistatic systems is limiting since the range (distance) and speed axes can only be defined for a single and arbitrary vectorial direction, whereas a key feature of multistatic systems is the ability to resolve targets based on position and velocity vectors. Therefore, a pair of new plots is introduced, which are referred to as the ‘position ambiguity plot’ (PAP) and ‘velocity ambiguity plot’ (VAP). The PAP is a snapshot of the position ambiguity response over the entire two-dimensional plane for a fixed hypothesized velocity, while the VAP is a snapshot of the response for all possible velocity vectors on polar axes for a fixed hypothesized target position.

Figure 3 shows the PAP and VAP for the same monostatic topology as above. In this case, the PAP response is plotted for the case when the hypothesized velocity vector equals the actual target velocity (in this case, 20 m/s in the South direction), and the axes limits are set to ±50 m from the true target position. Since the monostatic detector can natively resolve only in range (not angle), the PAP shows a section of an equi-range ambiguity circle with focus at the location of the radar \( (500,0) \) and intercepting the actual target position \( (500,500) \). The half-power thickness of the circle equals the nominal range resolution of 3m. Range ambiguity sidelobes are not clearly visible as the response power is plotted on a linear rather than logarithmic scale.

The VAP response is plotted for the case when the hypothesized target position equals the actual position \( (500,500) \). The VAP magnitude axis limits are 0 and 75 m/s while the angle ranges from 0 to 360 degrees. Since a monostatic detector can only resolve radial velocity, the response peaks at 20 m/s at 270 degrees (the actual target velocity), and is
also maximal for all velocity vectors where the component of that vector in the radial (270 degree) direction is 20 m/s (e.g. approximately 40 m/s at 210 degrees). The half-power width of this ambiguity ‘ridge’ equals the nominal velocity resolution of 30 m/s.

Figure 4 shows the PAP and VAP for the bistatic topology shown in Figure (1b). Here, the target has the same parameters as before, but the transmitter and receiver are positioned such that the target is close to, and has velocity vector orthogonal to, the bistatic baseline. This response confirms the well-known highly ambiguous nature of bistatic radar in this topology.

Figure 3: Monostatic position (PAP) and velocity (VAP) ambiguity plots
Having introduced these new plots, they can now be applied to multistatic systems where their utility for demonstrating target resolution based on position and velocity vectors will become evident. Three multistatic topologies are shown in Figure 5. The 'two-receiver' topology in Figure 5a is equal to the bistatic case above with the addition of a second receiver positioned at \{500,0\}. The 'sparse-array' topology in Figure 5b and the 'ring' topology in Figure 5c are chosen to demonstrate the nature of multistatic ambiguity where the receivers are closely spaced and widely spaced, respectively.
Figures 6 and 7 show the PAP and VAP for the coherent (Equation 12) and incoherent (Equation 13) ambiguity functions respectively for the two-receiver topology, followed by those for the sparse-array topology in Figures 8 and 9. The spacing between each of the 21 receivers in the sparse array is 10 m, giving a total effective aperture length of 200 m, and the single transmitter is co-located with the central element of the receiver array at \( \{500,0\} \). The axes limits of the PAPs from Figure 8 onwards are reduced to \( \pm 15 \) m to show the peak response clearly. Finally, Figures 10 and 11 show the PAP and VAP for the ring topology. The distance from the target to each of the 21 receivers is 500 m, and the transmitter is co-located with the receiver at \( \{500,0\} \).
Figure 6: PAP and VAP for two-receiver topology, coherent ambiguity function
Figure 7: PAP and VAP for two-receiver topology, incoherent ambiguity function

Figure 8: PAP (zoomed) and VAP for sparse-array topology, coherent ambiguity function
Figure 9: PAP (zoomed) and VAP for sparse-array topology, incoherent ambiguity function
Figure 10: PAP (zoomed) and VAP for ring topology, coherent ambiguity function

Figure 11: PAP (zoomed) and VAP for sparse-array topology, incoherent ambiguity
function
IX. Analysis of ambiguity responses

A qualitative summary of the nature of the ambiguity response for each multistatic detector and topology combination is shown in Table 2. Comparison of Figures 4, 6 and 7 shows that the inclusion of a single additional receiver to a bistatic radar system, sited well away from the existing baseline, yields some improvement in ambiguity when the target is close to the original bistatic baseline. However, significant ambiguous response exists over a much larger area of the plotted position and velocity planes than for the monostatic case in Figure 3. The PAP for the incoherent detector essentially demonstrates the power sum of the ambiguity response of the two constituent bistatic pairs; that of the coherent detector has a similar general appearance but the amplitude of the main ambiguity ridge varies rapidly over space due to constructive and destructive phasing between the two coherently summed responses. The VAPs are similar in both cases, and a better defined ambiguity ridge is present compared to the bistatic case since the actual target velocity direction is towards the second receiver.

In the case of the multistatic sparse-array topology, the coherent PAP in Figure 8 shows a well-defined central response at the actual target position surrounded by a series of ambiguity spikes. The periodic nature of these spikes may be considered similar to grating lobes in standard antenna array theory, caused by the spacing between receiver 'elements' being much larger than the RF wavelength. However, it should be borne in mind that the ambiguity response shown is dependent on the transmitted waveform as well as the antenna topology, and that the hypothesized positions are all in the near-field of the effective sparse array. Therefore, there is a stereoscopic focusing effect that causes the central peak to be somewhat narrower than the equivalent monostatic range resolution.
in the y-dimension, and very narrow in the x-dimension due to the large effective array aperture length. On the other hand, while the PAP for the incoherent detector in Figure 9 has a definite peak at the actual target position, there is significant ambiguity approximately following the iso-range curve. Here, since the response of each receiver is summed on a power basis, the array beamforming effect does not exist, and the close-in response approximates that of a standard monostatic radar in the same location. The VAPs are similar for both detectors, with poor velocity vector angle ambiguity that approaches the monostatic response, albeit possessing definite maxima at 270 degrees.

Finally, in the case of the ring topology where the receivers are more widely spaced, the coherent PAP in Figure 10 shows a well-defined, narrow peak at the actual target position with half-power width of approximately $\lambda/2$ (1.5 cm). Although not clearly visible in the figure, the maximum sidelobe is -4.8 dB and multiple sidelobes exist in the region surrounding the main peak in the range -7 to -10 dB. Such high sidelobes are a typical feature of the response of sparsely distributed arrays. The incoherent PAP in Figure 11 also shows a central peak, however its half-power width is much larger (around 3 m) since it arises from the power sum of each bistatic ambiguity response. The velocity diagrams both show a single peak centered at the point defining the actual target velocity. Therefore, this topology is capable of effective target resolution based on both angle and magnitude of the velocity vectors.

<table>
<thead>
<tr>
<th>Topology</th>
<th>Coherent detector</th>
<th>Incoherent detector</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Position</td>
<td>Velocity</td>
</tr>
</tbody>
</table>

Table 2: Qualitative summary of the example multistatic ambiguity responses
X. Discussion

From these results, it is possible to make some comments about the nature of multistatic ambiguity with different topologies using the coherent and incoherent detectors. Firstly, all the multistatic examples shown demonstrate some degree of target resolution capability in two dimensions, independently of individual antenna beam patterns, which is an immediate advantage over monostatic radars. However, the most noteworthy results are obtained when a relatively large number of widely-spaced receivers are used; the total ambiguity response results from a combination of time-of-arrival ambiguity (determined by the transmitted waveform) and spatial processing ambiguity, which depends significantly on the multistatic topology and the detector (and, therefore, the target model).

Figures 8 and 10 show that a multistatic system performing coherent detection can produce ambiguity responses that allow two-dimensional or three-dimensional spatial resolution that is far better than the nominal range resolution implied by the bandwidth of the waveform. In these topologies, where the target is (respectively) in the near-field of, and encircled by, the effective array aperture, the spatial array processing performed by
the coherent detector dominates the time-of-arrival processing related to the transmitted signal. The resolution of this spatial processing is fundamentally limited only by the RF wavelength, however in practice the effective array may be very sparse and irregular, and a wide bandwidth transmitted signal may allow the resulting spatial ambiguity (equivalent to sidelobes or grating lobes in the 2D plane) to be mitigated. This high resolution result is fundamentally in agreement with the predicted response of MIMO radar with 'spatial multiplexing' gains [23], except in this case only a single transmitter is used. Radar systems performing incoherent detection can also achieve good three-dimensional localization in space when the receivers are widely spaced, but the width of the main response peak in any dimension is limited to the nominal monostatic range resolution determined by the waveform bandwidth.

Resolution based on target velocity vectors is strongly determined by the topology, but less dependent on the type of detector, as demonstrated by Figures 10 and 11. Since each bistatic pair ‘sees’ only the components of the velocity vector in the direction of each transmitter and receiver (see Equation 11), the receivers must be widely dispersed around the target to effectively resolve the direction of travel.

The detectors defined in Section IV are optimal in the sense of statistical detection probability (Neyman-Pearson criterion), but in the multistatic case are not necessarily optimal in terms of target resolution. For example, in the multistatic topology in Figure 5a, it is apparent that the ambiguity demonstrated in Figures 6 and 7 could be improved by simply ignoring the signal from the first receiver, which is highly ambiguous since the target is close to the bistatic baseline it forms with the transmitter. In general, additional amplitude weighting terms might be applied prior to multistatic summation that result in
suboptimal detection performance but minimize ambiguity in certain regions of the position and velocity planes to suit particular applications [22].

It is assumed in the derivation of the multistatic ambiguity responses that all parameters in Equations 9 and 10, except for the received signals $X_i(t)$, are a-priori known by the detectors. In the incoherent case, the noise power $N_i$ and mean signal amplitude $\bar{A}_i$ and energy $\bar{E}_i$ of the fluctuating signals might in practice be adaptively estimated in real-time or by using training sequences over multiple observations. In the coherent case, the relative phase $\varphi_{ni}$ of the correlated fluctuations must also be estimated. In many cases where the receivers are closely spaced, the phase of fluctuations as observed by each receiver will be the same such that all $\varphi_{ni}$ can be assumed to equal zero. In other cases, for example where widely spaced receivers observe different aspects of a non-fluctuating target, the relative phases are fixed but unknown, and adequate estimates of $\varphi_{ni}$ might be obtainable in real-time by trial-and-error search for maximum coherent gain.

It is somewhat unfortunate that, in the widely spaced topology where the best resolution can be obtained using coherent processing, fluctuations may be mutually independent for many complex targets. However, note that since in general multistatic radar does not require mechanical antenna scanning to survey the coverage space, the time between observation periods may be much shorter than that implied for a Swerling I target where fluctuations are assumed independent from scan to scan. Therefore, it is feasible that for some targets the fluctuations may be temporally correlated over an adequately long period for parameters $A_{ni}$ and $\varphi_{ni}$ to be estimated and coherent detection performed as if the target were non-fluctuating.
However, if indeed the observed fluctuations are independently fluctuating between observation periods, the loss of coherent SNR gains and potential for high resolution may be compensated for by diversity gains in detection performance due to the avoidance of fluctuation nulls at all receivers simultaneously. It is clear that, in many cases, the choice of detector will be dictated not by the user but by the available multistatic topology and the spatial nature of fluctuations from the target being observed.

Finally, no matter whether coherent or incoherent detection is used, it is necessary for all transmitters and receivers in the multistatic system to be mutually time synchronized for the measured propagation delays to be matched. Further, the relative locations of all transmitters and receivers must be known or estimated by calibration. In the coherent case, the maximum allowable calibration error may be much less than one RF wavelength. In addition, the signal processing required to implement the detection algorithms may be intensive due to the nonlinear relationships between the received signals and the measured parameters. These requirements separate such multistatic radars from traditional systems where information is shared at a plot or track level, and achieving them is one major challenge to practical deployment [25].

**XI. Conclusions**

Ambiguity functions for spatially coherent and incoherent multistatic radar have been derived on the basis of the corresponding optimal detectors. These expressions provide a powerful tool to demonstrate the capabilities and response of such systems. It has been shown that the ambiguity response is highly dependent on the multistatic topology and the type of detector, and therefore on the spatial coherency of target fluctuations. Various
examples have been used to demonstrate the effect of adding a single additional receiver to a bistatic system, and the performance of systems with a large number of receivers that are closely or widely spaced. It was shown that target parameters can be resolved in two-or three-dimensions, and in certain cases extremely high resolution target localization is possible. There are many possibilities for future work on analyzing the ambiguity response arising from other multistatic topologies and choice of transmitted waveform. The practical realization of such systems depends upon the development of suitable methods for achieving the necessary time and phase coherency across the entire multistatic network.

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References


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